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Technical Report No. 32-41

Selenographic Coordinates

B. E. Kalensher

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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SELENOGRAPHIC COORDINATES

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ABSTRACT

The Moon's triaxial, gravitational potential and its complex, rotary motion (libration) are incorporated in the equations of motion of a space vehicle in the near vicinity of the Moon. A transformation is derived between the space-fixed coordinate system, centered in the Moon, in which the vehicle's motion is computed, and a coordinate system fixed in the lunar body, i.e., a coordinate system which rotates with the Moon. Finally, an expression is derived for computing the velocity of the space vehicle in the Moon-fixed coordinate system.

I. INTRODUCTION

Two lunar missions which will provide important scientific information about the Moon are the orbiting of an instrumented probe around the Moon and the landing of a vehicle at some designated spot on the Moon's surface. One prerequisite for the success of either mission is that the standard, or "preflight," trajectory be simulated with sufficient realism when the probe is close to the Moon. Sufficient realism can be achieved by incorporating the Moon's triaxial (gravitational) potential and its complex, rotary motion in the equations of motion of the probe. This has been done in the present paper. The resulting set of equations contains a complete and accurate description of the Moon's motion, as well as an up-to-date representation of the lunar triaxial potential.[1]

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[1] In order to implement the *Ranger* Program (semi-soft lunar impact) currently under study at the Jet Propulsion Laboratory, Eq. (5) - (8), (12) - (15), (18) - (42), (44a), (51) - (56), and (58) have been programmed on the 704 digital computer. The nutation in longitude ψ and the nutation in obliquity E — which are the smallest angles computed in the program — agree with the corresponding values in the *American Ephemeris and Nautical Almanac* to within 10^{-6} deg. This agreement is sufficient for an accurate simulation of the Moon's motion.

II. TRIAXIAL POTENTIAL

As long as the space probe remains at large distances from the Earth, Moon, or planets, the gravitational potential of these bodies can be written simply as K/r. If the probe moves closer than approximately 4 radii from the center of the body, K/r will no longer be valid, since probably the primary mass will not be a homogeneous sphere. For example, observations of the Moon over the past one hundred years indicate that the lunar body is a nonhomogeneous, triaxial ellipsoid; i.e., the Moon has only three principal axes of inertia (the Moon is an ellipsoid, but not one of revolution). In order to accurately describe a mission in which an instrumented probe is either orbited around the Moon (satellite) or landed on the Moon's surface (soft impact), account must be taken of the true potential V. The gravitational potential (per unit mass) at a point P distant r from the center of mass O of any rigid body of mass M is

$$V = -G\left(\frac{M}{r} + \frac{A+B+C-3I}{2r^3}\right) + O\left(\frac{1}{r^4}\right) \tag{1}$$

where G is universal gravitation constant, I is moment of inertia of M about OP, and A, B, C are moments of inertia of M about the three principal axes of inertia. (Any rigid body has at least three orthogonal, principal axes of inertia.) If we construct a rectangular coordinate system, fixed in M, with origin at O and axes coincident with the three principal axes of M, then

$$I = A\left(\frac{x'}{r}\right)^2 + B\left(\frac{y'}{r}\right)^2 + C\left(\frac{z'}{r}\right)^2$$

where x', y', z' are the coordinates of P in this system and A, B, C are taken about x', y', z', respectively. Also, $r^2 = x'^2 + y'^2 + z'^2$. By making use of (1), the exact equations of motion of the probe in the vicinity of the Moon can be derived. For all practical purposes, the higher order terms in $1/r^4$ can be dropped.

III. EQUATIONS OF MOTION

The proximity of the probe to the Moon makes it advantageous to compute the probe's motion in a coordinate system centered at the Moon. Although the system x', y', z' is so centered, computing in it would be unwise, since it is fixed in the Moon and consequently would partake of the Moon's intricate rotation. A convenient coordinate system to use is one with origin at the center of the Moon, x axis pointing in the direction of the mean[1] vernal equinox[2] of 1950.0 (i.e., the beginning of the Besselian year of 1950) and the y axis so chosen as to make the xy plane parallel to the mean equator of the Earth as of 1950.00. The z axis of this right-handed system will point in the direction of the "mean" spin axis of the Earth. Two reasons for selecting this particular system are (1) the directions of the coordinate axes are fixed in space, and (2) the positions of the Moon (and hence the Earth), Sun, and planets are available in this system (when centered at the Earth).

Let N equal the total number of celestial bodies in a given region of space. Denote the individual bodies by $i = 1, 2, \cdots, N$. Construct a right-handed, rectangular coordinate system (x, y, z) with origin at the center of mass of the Nth body ($i = N$) and axes fixed in an arbitrary direction, relative to the "fixed" stars. Let $i = 1$ denote the probe, and assume that the gravitational potential of $i = 2, 3, \cdots, N$ is given by Eq. (1). In addition to the gravitational forces acting on $i = 1$, assume that it is also acted upon by a thrust $\mathbf{F}$. Then the equations of motion of the probe, in the system (x, y, z), are:

[1] "Mean" signifies that the Earth's nutation has been ignored.

[2] The vernal equinox is that point on the celestial sphere (relative to the fixed stars) in the constellation Pisces at which the Earth's equator and the ecliptic (orbital plane of the Earth) appear to intersect. (The autumnal equinox is the other intersection of the ecliptic and equator, 180 deg from the vernal equinox, in the constellation Virgo). Since the Earth's orbit and equatorial plane are continually changing (by small amounts, relative to the fixed stars), the position of the equinox is also continually changing.

$$
\left.
\begin{aligned}
\ddot{x}_1 &= \sum_{i=2}^{N} G \left(H_i \frac{x_1 - x_i}{r_{i1}} - \frac{3}{r_{i1}^5} \, \mathbf{p} \cdot \mathbf{i} \right) - \sum_{i=1}^{N-1} \frac{G m_i}{r_i^3} x_i + \frac{F_x}{m_1} \\
\ddot{y}_1 &= \sum_{i=2}^{N} G \left(H_i \frac{y_1 - y_i}{r_{i1}} - \frac{3}{r_{i1}^5} \, \mathbf{p} \cdot \mathbf{j} \right) - \sum_{i=1}^{N-1} \frac{G m_i}{r_i^3} y_i + \frac{F_y}{m_1} \\
\ddot{z}_1 &= \sum_{i=2}^{N} G \left(H_i \frac{z_1 - z_i}{r_{i1}} - \frac{3}{r_{i1}^5} \, \mathbf{p} \cdot \mathbf{k} \right) - \sum_{i=1}^{N-1} \frac{G m_i}{r_i^3} z_i + \frac{F_z}{m_1}
\end{aligned}
\right\}
\qquad (2)
$$

where

$$
H_i = \frac{-m_i}{r_{i1}^2} - 3/2 \, \frac{A_i + B_i + C_i}{r_{i1}^4} + \frac{15}{2} \, \frac{1}{r_{i1}^4} \left[A_i \left(\frac{x_i'}{r_{i1}} \right)^2 + B_i \left(\frac{y_i'}{r_{i1}} \right)^2 + C_i \left(\frac{z_i'}{r_{i1}} \right)^2 \right]
$$

$$
\mathbf{P} = A_i x_i' \, \mathbf{i}_i' + B_i y_i' \, \mathbf{j}_i' + C_i z_i' \, \mathbf{k}_i'
$$

Here, x_i', y_i', z_i' are the coordinates of the probe measured in a rectangular coordinate system coincident with the principal axes in the ith body ($i = 2, 3, \cdots, N$), and $\mathbf{i}_i'$, $\mathbf{j}_i'$, $\mathbf{k}_i'$ are unit vectors along these respective axes; $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along x, y, z, respectively; m_i denotes the mass of the ith body; r_i is the distance between m_N and m_i ($r_N = 0$); r_{i1} is the distance between m_1 and m_i ($r_{11} = 0$). Finally,

$$\mathbf{r}_i = \mathbf{i}\,x_i + \mathbf{j}\,y_i + \mathbf{k}\,z_i \qquad (i = 1, 2, \cdots, N-1)$$

$$\mathbf{r}_{i1} = \mathbf{i}(x_1 - x_i) + \mathbf{j}(y_1 - y_i) + \mathbf{k}(z_1 - z_i)$$

$$= \mathbf{i}_i'\,x_i' + \mathbf{j}_i'\,y_i' + \mathbf{k}_i'\,z_i' \qquad (i = 2, 3, \cdots, N)$$

where

$$x_N = y_N = z_N = 0$$

In order to apply the above formalism to the Moon, we let $N = 4$ and adopt the following convention: Let,

i = 1 denote the probe

2 denote the Earth

3 denote the Sun

4 denote the Moon

Since the probe will be in the vicinity of the Moon, the Earth and Sun can be regarded as perfect homogeneous spheres (i. e., point masses), so that $A_2 = B_2 = C_2$, $A_3 = B_3 = C_3$. Also, the mass of the probe (m_1) will be negligible compared with the masses of the Earth, Sun, and Moon. Therefore, Eq. (2) become:

$$\left.\begin{aligned}
\ddot{x}_1 &= \sum_{i=2}^{3} \frac{G m_i}{r_{i1}^3}(x_i - x_1) - \sum_{i=2}^{3} \frac{G m_i}{r_i^3}\, x_i + \frac{F_x}{m_1} + G\left[H\frac{x_1}{r_1} - \frac{3}{r_1^5}(A\,a_{11}\,x_1' + B\,a_{21}\,y_1' + C\,a_{31}\,z_1')\right]\\
\ddot{y}_1 &= \sum_{i=2}^{3} \frac{G m_i}{r_{i1}^3}(y_i - y_1) - \sum_{i=2}^{3} \frac{G m_i}{r_i^3}\, y_i + \frac{F_y}{m_1} + G\left[H\frac{y_1}{r_1} - \frac{3}{r_1^5}(A\,a_{12}\,x_1' + B\,a_{22}\,y_1' + C\,a_{32}\,z_1')\right]\\
\ddot{z}_1 &= \sum_{i=2}^{3} \frac{G m_i}{r_{i1}^3}(z_i - z_1) - \sum_{i=2}^{3} \frac{G m_i}{r_i^3}\, z_i + \frac{F_z}{m_1} + G\left[H\frac{z_1}{r_1} - \frac{3}{r_1^5}(A\,a_{13}\,x_1' + B\,a_{23}\,y_1' + C\,a_{33}\,z_1')\right]
\end{aligned}\right\} \quad (3)$$

where

$$H = \frac{-m_4}{r_1^2} - \frac{3}{2}\,\frac{A+B+C}{r_1^4} + \frac{15}{2}\,\frac{1}{r_1^4}\left[A\left(\frac{x_1'}{r_1}\right)^2 + B\left(\frac{y_1'}{r_1}\right)^2 + C\left(\frac{z_1'}{r_1}\right)^2\right]$$

Here, x_1', y_1', z_1' now denote the coordinates of the probe in the coordinate system x', y', z', which is centered at the Moon and aligned with the Moon's principal axes of inertia. Also,

$$r_1^2 = x_1^2 + y_1^2 + z_1^2 = x_1'^2 + y_1'^2 + z_1'^2$$

The quantities a_{ij} define the transformation between x', y', z' and the coordinate system x, y, z, which we now identify with that of 1950.0 (described above). Thus,

$$\left.\begin{aligned} x_1' &= a_{11}\,x_1 + a_{12}\,y_1 + a_{13}\,z_1 \\ y_1' &= a_{21}\,x_1 + a_{22}\,y_1 + a_{23}\,z_1 \\ z_1' &= a_{31}\,x_1 + a_{32}\,y_1 + a_{33}\,z_1 \end{aligned}\right\} \qquad (4)$$

IV. SELENOGRAPHIC COORDINATES

When the Moon was still in a liquid state, the tidal waves created on its surface by the Earth's gravitational field distorted the Moon's shape and transformed its rotational energy into heat energy (by friction), thereby reducing its axial rotation and eventually forcing it to present the same face to the Earth. As the Moon solidified, it acquired a permanent disfiguration, chiefly in the form of a bulge on the side facing the Earth. This deformation caused the Moon to undergo small, pendulous oscillations, which still exist and which are called "physical librations." The principal axes of inertia and hence the axes x', y', z' are assumed (by the astronomers) to lie, respectively, through the center of the bulge toward the Earth, at right angles to this in a direction opposite to the Moon's orbital motion, and mutually perpendicular to the aforementioned (refer to Fig. 1). Thus, the x', y' plane is the Moon's equatorial plane and the z' axis coincides with the Moon's axis of rotation. The x' axis passes through the Sinus Medii (Central Bay) on the lunar surface.

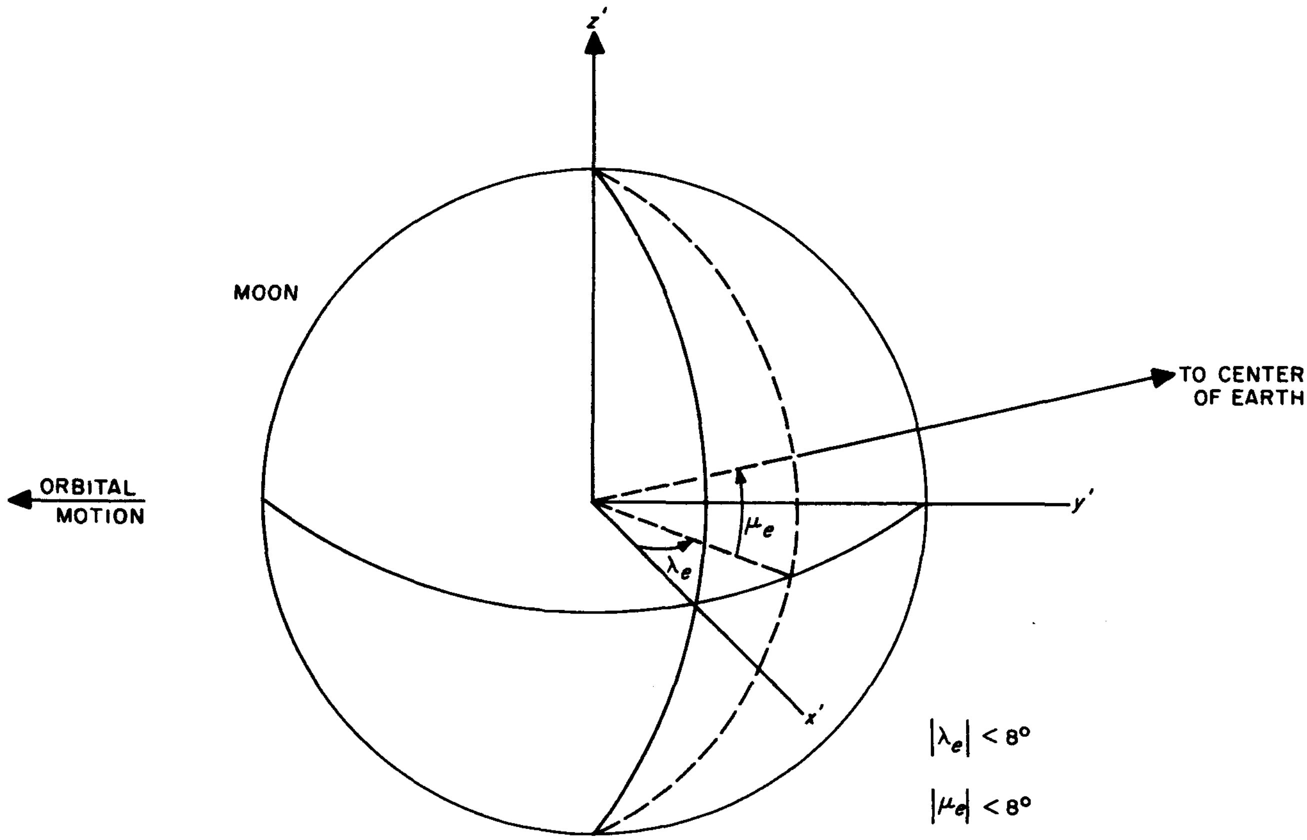

Fig. 1. Definition of selenographic latitude and longitude of Earth

The orientation of the Moon as it orbits the Earth is illustrated in Fig. 2 and 3. Here, it is assumed that the physical librations are zero. (The amplitude of these librations never exceeds $0°.04$.) Figure 2 is a view of the Earth–Moon system as viewed from a point in the ecliptic (which is normal to the paper).

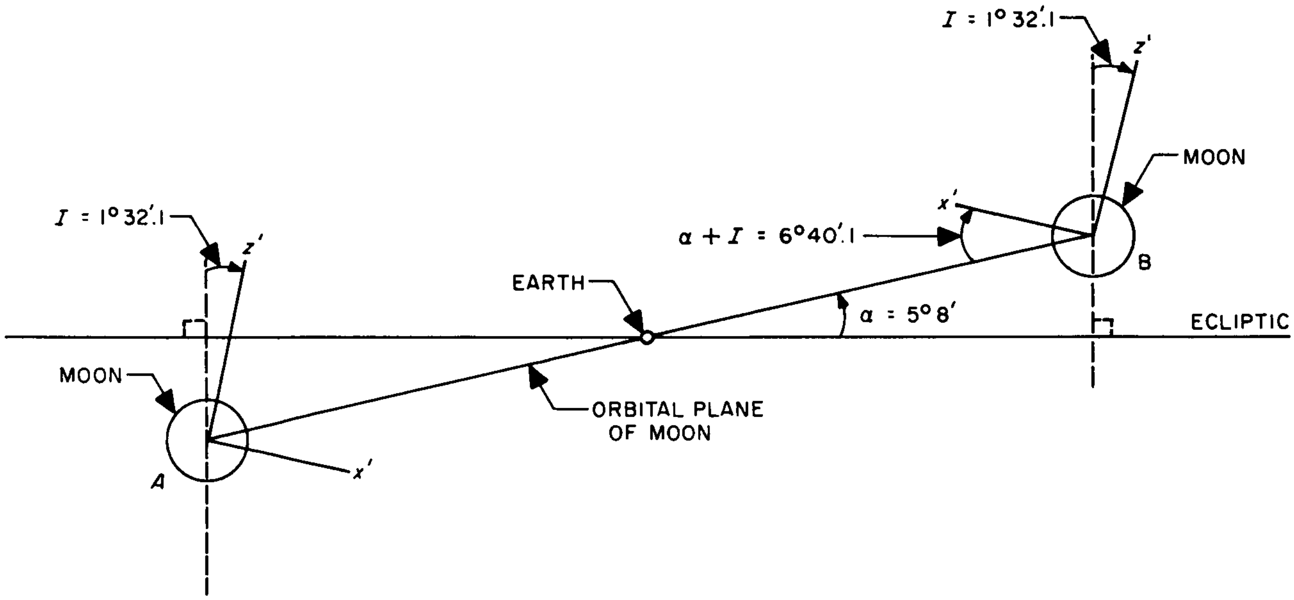

Fig. 2. Earth–Moon system viewed from ecliptic

The inclination of the Moon's orbit to the ecliptic α may vary as much as $0°.30$, whereas $\delta I < 0°.043$. Since the inclination of the Moon's equator to its orbital plane, $\alpha + I$, is not zero ($\sim 6°40'$), an observer on the Earth would, at one time, see the north pole of the Moon (position A in Fig. 2) and half a month later see the south pole (position B). This apparent oscillation of the Moon's poles is called the "optical libration in latitude." Figure 3 shows the Earth–Moon system as viewed from a point above the Moon's orbital plane. Since the Moon moves in an ellipse around the Earth, its orbital angular velocity is not constant (the Moon's spin angular velocity is practically constant). Therefore, to an observer on the Earth, the Moon would appear to oscillate about its spin axis z' (refer to Fig. 3). This apparent oscillation is called the "optical libration in longitude." The optical librations in latitude and longitude vary in magnitude between, approximately, ± 8 deg.

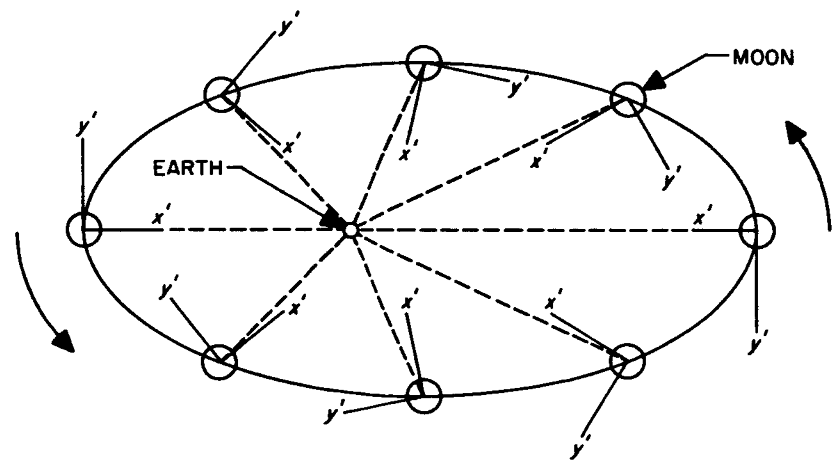

Fig. 3. Earth – Moon system viewed from above orbital plane

Like the optical librations, the physical librations occur in both longitude and latitude. It is evident from Fig. 2 and 3 that the optical librations have a period of a month (approx 28 days). The physical libration in longitude, however, has a period of one year, and the physical libration in latitude has a period of approximately six years. The physical librations for the years 1956 – 1961 are shown in Fig. 4. Even though these librations are negligible compared with the optical librations, they are still essential to an accurate simulation of the Moon's motion. Thus, a libration of only 0°.02 results in a displacement of 600 meters on the Moon's surface. This distance might become critical in the event of a vehicle soft-landing at a designated point on the lunar surface.

In the absence of all librations, the x' axis would always point to the center of the Earth. Its deviation from this direction is due to the combined effect of both optical and physical librations. The motion of the x' axis can be described by means of the Earth's selenographic coordinates. The Earth's selenographic coordinates are the position coordinates of the Earth measured in a coordinate system fixed in the Moon. Here, the Moon-fixed system is simply x', y', z'. The Earth's selenographic latitude μ_e is the angle that the line connecting the centers of Moon and Earth makes with the $x' y'$ plane (refer to Fig.1). The Earth's selenographic longitude λ_e is the angle between the $x' z'$ plane and the plane containing the Earth – Moon line and the z' axis; λ_e is measured in the $x' y'$ plane. The positive directions of μ_e and λ_e are indicated in Fig. 1. If the Moon did not librate then $\mu_e \equiv \lambda_e \equiv 0$.

The angles μ_e, λ_e are computed in terms of the observable quantities: i, Λ, Ω', A, ϕ. Here, A and ϕ are the right ascension and declination, respectively, of the Moon; i, Λ, Ω' are Euler angles describing the orientation

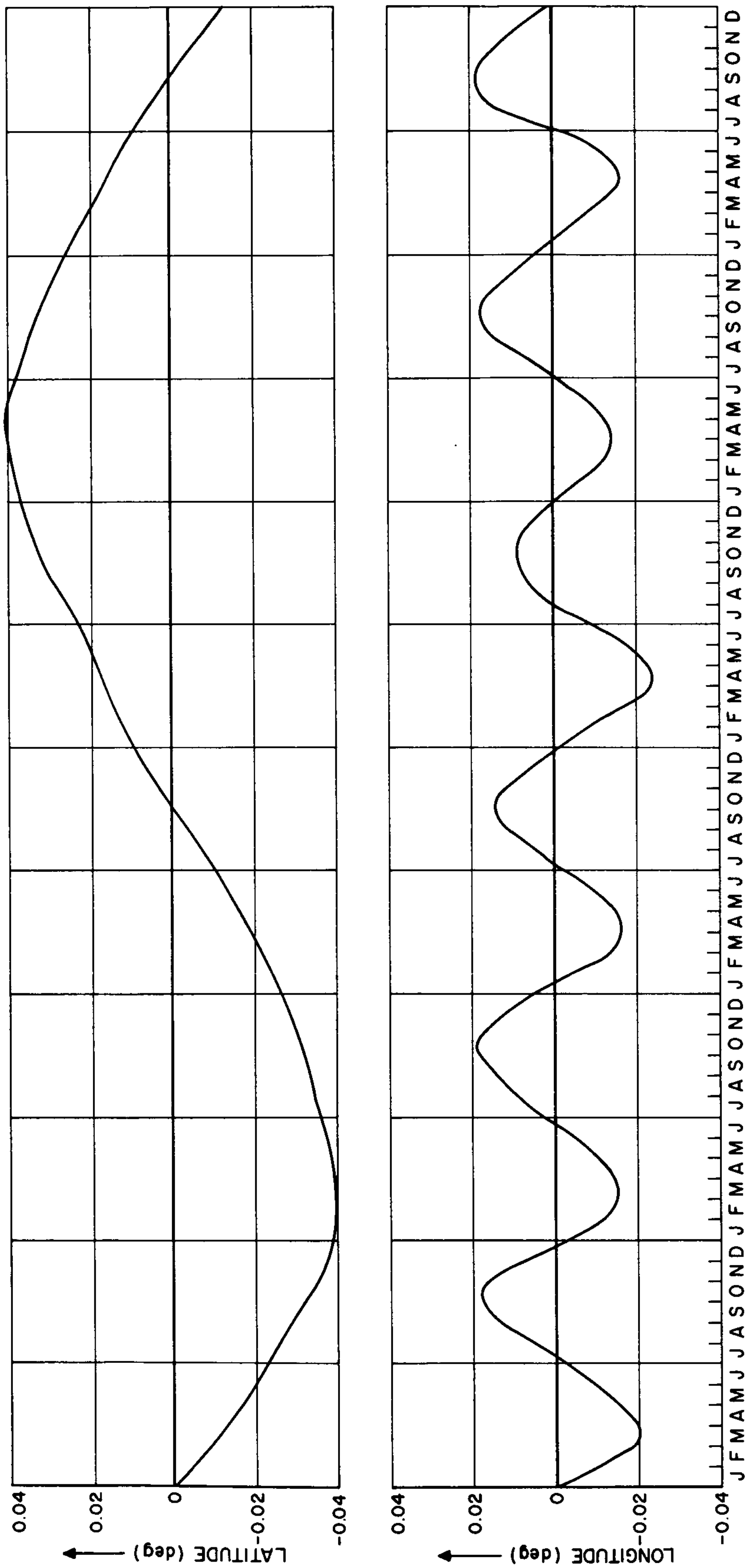

Fig. 4. Physical librations of the Moon

of x', y', z' relative to the set of axes $\tilde{x}$, $\tilde{y}$, $\tilde{z}$. Here, $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ is similar to the coordinate system x, y, z of 1950.0 (described above), except that the positive $\tilde{x}$ axis points in the direction of the true (the Earth's nutation is not ignored) equinox[3] of date (i. e., the position of the equinox at any specified instant, or date) and the $\tilde{x}\,\tilde{y}$ plane is parallel to the Earth's true equator of (the same) date (refer to Fig. 5).

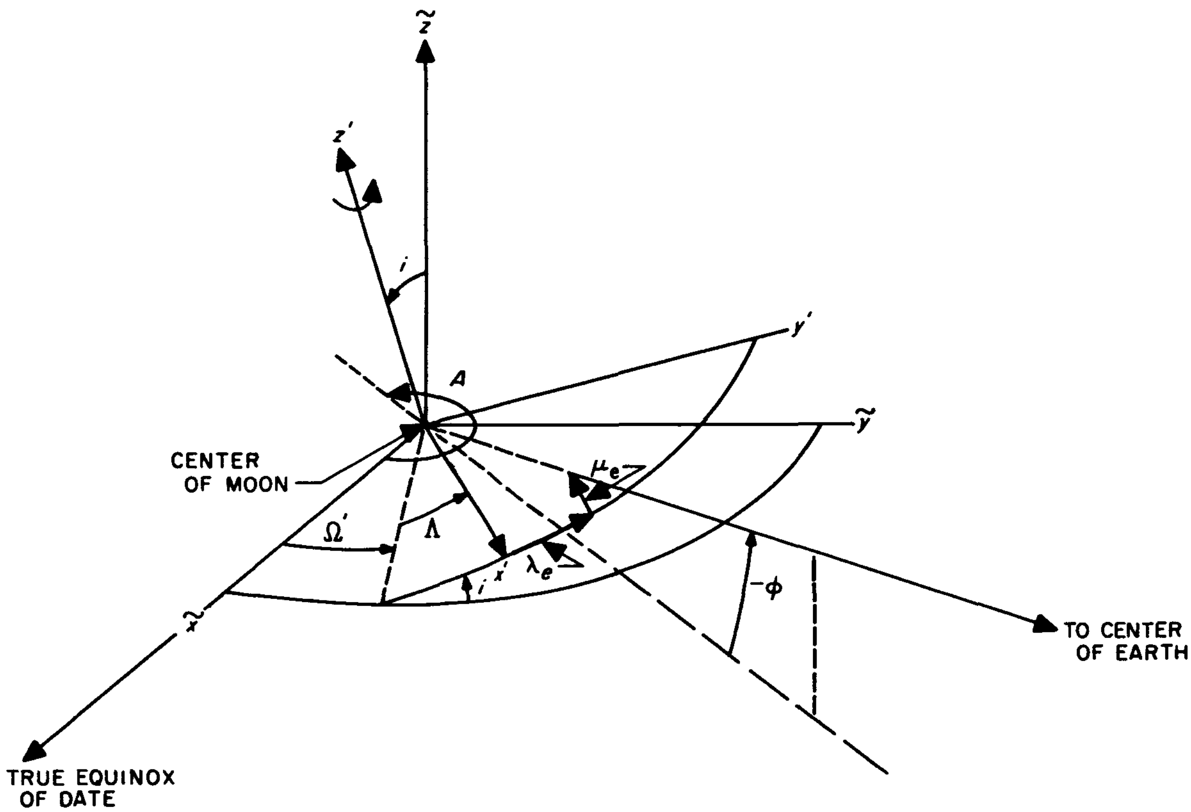

Fig. 5. Definition of i, Ω', Λ, A, ϕ

It can be shown that the Earth's selenographic latitude and longitude are given by

$$\sin \mu_e = \sin i \sin(A - \Omega') \cos \phi - \cos i \sin \phi \tag{5}$$

[3]The vernal equinox.

$$\lambda_e = \sin^{-1}\left[\frac{-\sin(A - \Omega')}{\sin\psi}\right] + \sin^{-1}\left(\frac{\tan\mu_e}{\tan\psi}\right) - \Lambda \tag{6}$$

where $\cos\psi = -\sin i \cos(A - \Omega')$. It is always true that $|\mu_e| < 8$ deg, $|\lambda_e| < 8$ deg. Figure 6 shows the selenographic coordinates $R\,\mu_e \approx z'$ (R = mean radius of Moon = 1738.11 km) and $R\,\lambda_e \approx y'$ plotted on the Moon's surface for May 1960.

The landing of a probe at a designated spot on the Moon's surface requires that the probe's selenographic coordinates be known. The probe's selenographic latitude μ_p and longitude λ_p are defined in Fig. 7. Here,

$$\sin\mu_p = \frac{z_1'}{\left(x_1'^2 + y_1'^2 + z_1'^2\right)^{\frac{1}{2}}} \tag{7}$$

$$\sin\lambda_p = \frac{y_1'}{\left(x_1'^2 + y_1'^2\right)^{\frac{1}{2}}}, \quad \cos\lambda_p = \frac{x_1'}{\left(x_1'^2 + y_1'^2\right)^{\frac{1}{2}}} \tag{8}$$

where x_1', y_1', z_1' are computed from Eq. (4).

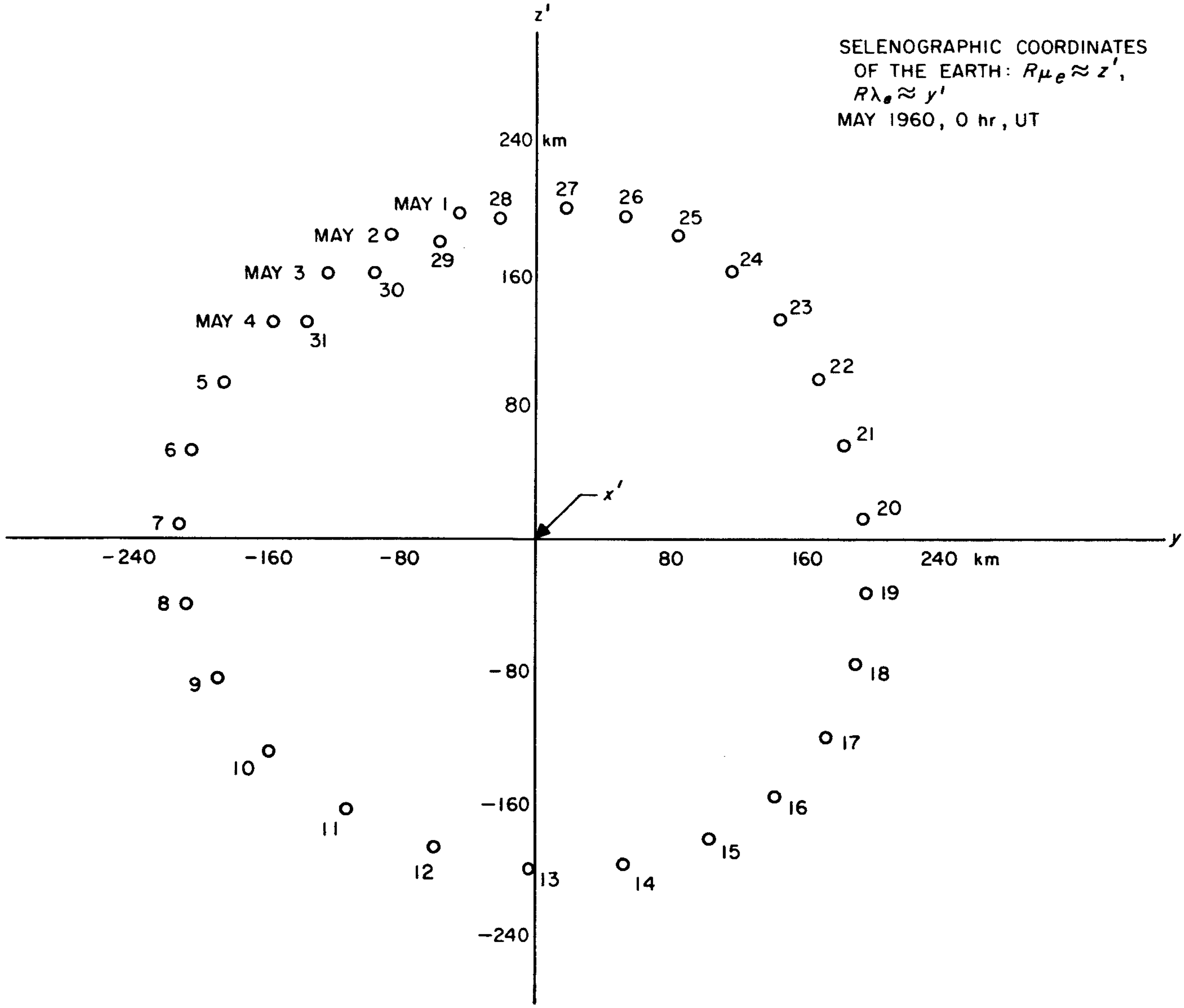

Fig. 6. Selenographic coordinates of the Earth: $R\mu_e \approx z'$, $R\lambda_e \approx y'$. May 1960, 0 hr, UT

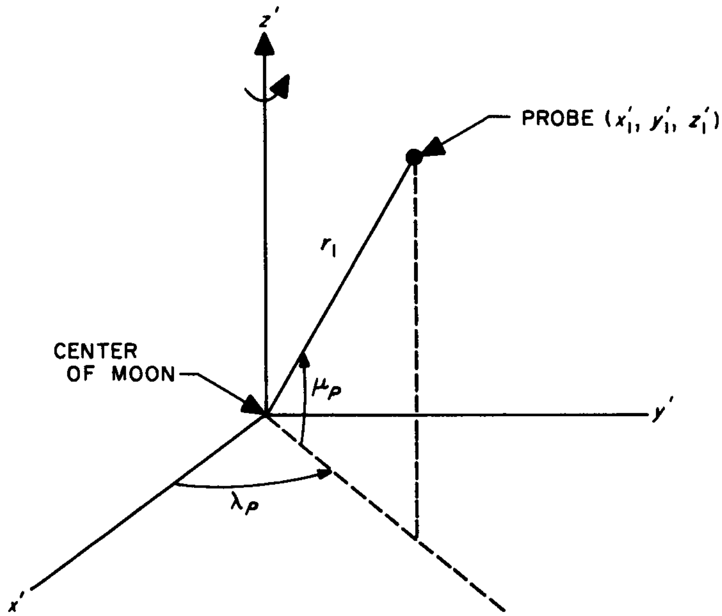

Fig. 7. Selenographic coordinates of probe

V. THE TRANSFORMATION COEFFICIENTS a_{ij}

The coefficients a_{ij} in Eq. (4) describe the transformation from the system x, y, z of 1950.0, previously defined, to the Moon-fixed system x', y', z'. This transformation is the net result of three separate coordinate transformations, which are given below. First, it is necessary to introduce a fourth system $\bar{x}$, $\bar{y}$, $\bar{z}$, which is similar to $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, except that $\bar{x}$ points to the mean equinox of date and the $\bar{x}\,\bar{y}$ plane is parallel to the Earth's mean equator of date. We let X' denote the column vector $\{x' \; y' \; z'\}$, and similarly for the other three systems. Then the transformations are:

$$\tilde{X} = a' \, X' \tag{9}$$

$$\bar{X} = \tilde{a} \, \tilde{X} \tag{10}$$

$$X = \bar{a} \, \bar{X} \tag{11}$$

where a', $\tilde{a}$, $\bar{a}$ are the transformation coefficient matrices. Combining (9), (10), (11) gives

$$X' = (\bar{a} \, \tilde{a} \, a')^{-1} \, X = a \, X \tag{12}$$

which is Eq. (4). In order to determine $a = (a_{ij})$, we must first investigate a', $\tilde{a}$, $\bar{a}$.

The transformation (9) is illustrated in Fig. 5 (if one ignores A, ϕ, μ_e, λ_e). Making use of spherical trigonometry, it is easy to show that

$$\left.\begin{aligned}
a'_{11} &= \cos\Omega' \, \cos\Lambda - \sin\Omega' \, \sin\Lambda \, \cos i \\
a'_{21} &= \cos\Omega' \, \cos\Lambda + \cos\Omega' \, \sin\Lambda \, \cos i \\
a'_{31} &= \sin\Lambda \, \sin i \\
a'_{12} &= -\cos\Omega' \, \sin\Lambda - \sin\Omega' \, \cos\Lambda \, \cos i \\
a'_{22} &= -\sin\Omega' \, \sin\Lambda + \cos\Omega' \, \cos\Lambda \, \cos i \\
a'_{32} &= \cos\Lambda \, \sin i \\
a'_{13} &= \sin\Omega' \, \sin i \\
a'_{23} &= -\cos\Omega' \, \sin i \\
a'_{33} &= \cos i
\end{aligned}\right\} \tag{13}$$

The inclination of the ecliptic to the Earth's equator is called the obliquity. The existence of a mean equator and true equator, of date, implies the existence of a mean obliquity $\bar{\epsilon}$ and a true obliquity $\tilde{\epsilon}$, of date. The difference, $\tilde{\epsilon} - \bar{\epsilon}$, is called the nutation in obliquity E.

$$\tilde{\epsilon} - \bar{\epsilon} = E \tag{14}$$

The above are illustrated in Fig. 8.

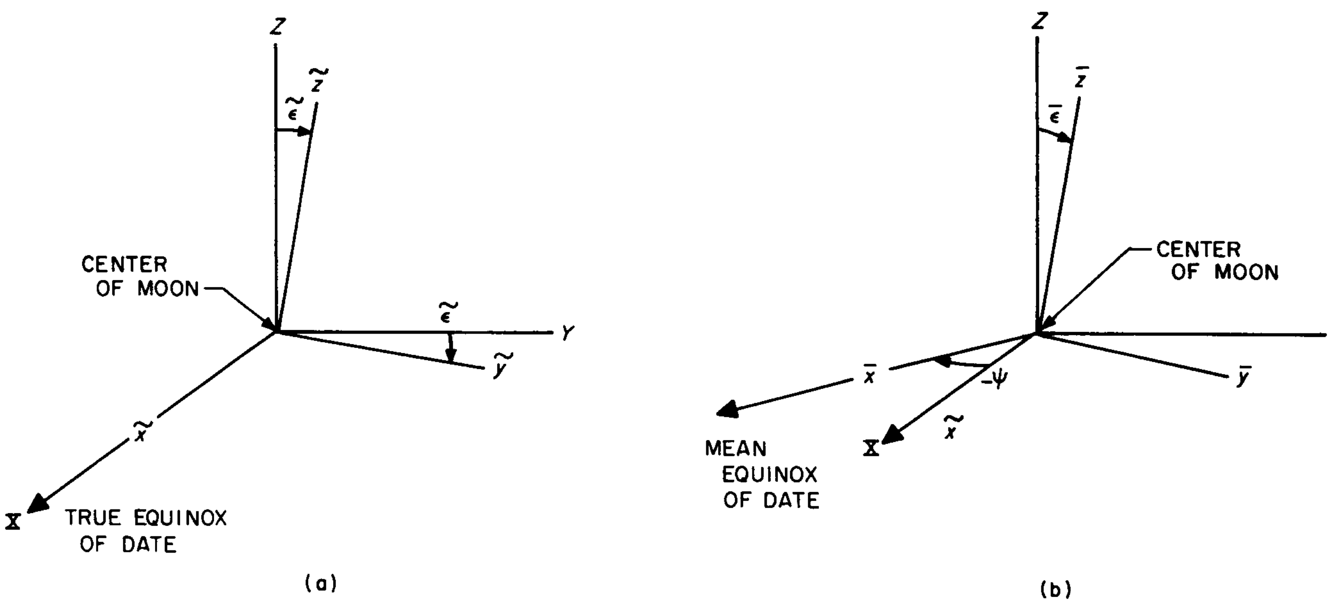

Fig. 8. Definition of mean and true obliquity

Here, the $X\ Y$ plane is parallel to the plane of the ecliptic. The angle ψ, which is measured in this plane, is called the nutation in longitude. If we denote the column vector $\{X\ Y\ Z\}$ by X_e, then

$$\tilde{X} = B\, X_e \qquad\qquad \bar{X} = C\, X_e$$

Combining the above gives

$$\bar{X} = C\, B^{-1}\, \tilde{X} = \tilde{a}\, \tilde{X} \tag{10a}$$

which is Eq. (10). The coefficients B_{ij} and C_{ij} can be derived from Fig. 8a and 8b, respectively. Thus

$$B_{11} = 1, \quad B_{12} = B_{13} = B_{21} = B_{31} = 0, \quad B_{22} = B_{33} = \cos \tilde{\epsilon},$$

$$B_{23} = -B_{32} = -\sin \tilde{\epsilon}$$

$$C_{11} = \cos \psi, \quad C_{12} = \sin \psi, \quad C_{13} = 0, \quad C_{21} = -\cos \bar{\epsilon} \sin \psi$$

$$C_{22} = \cos \bar{\epsilon} \cos \psi, \quad C_{23} = -\sin \bar{\epsilon}, \quad C_{31} = -\sin \bar{\epsilon} \sin \psi$$

$$C_{32} = \sin \bar{\epsilon} \cos \psi, \quad C_{33} = \cos \bar{\epsilon}$$

Since $\tilde{a} = C\,B^{-1}$, therefore

$$\left.\begin{aligned}
\tilde{a}_{11} &= \cos \psi \\
\tilde{a}_{12} &= \cos \tilde{\epsilon} \sin \psi \\
\tilde{a}_{13} &= \sin \tilde{\epsilon} \sin \psi \\
\tilde{a}_{21} &= -\cos \bar{\epsilon} \sin \psi \\
\tilde{a}_{22} &= \cos \bar{\epsilon} \cos \tilde{\epsilon} \cos \psi + \sin \bar{\epsilon} \sin \tilde{\epsilon} \\
\tilde{a}_{23} &= \cos \bar{\epsilon} \sin \tilde{\epsilon} \cos \psi - \sin \bar{\epsilon} \cos \tilde{\epsilon} \\
\tilde{a}_{31} &= -\sin \bar{\epsilon} \sin \psi \\
\tilde{a}_{32} &= \sin \bar{\epsilon} \cos \tilde{\epsilon} \cos \psi - \cos \bar{\epsilon} \sin \tilde{\epsilon} \\
\tilde{a}_{33} &= \sin \bar{\epsilon} \sin \tilde{\epsilon} \cos \psi + \cos \bar{\epsilon} \cos \tilde{\epsilon}
\end{aligned}\right\} \tag{15}$$

In going from the mean equator and equinox of date to the mean equator and equinox of epoch (i. e., a specified date), as exemplified by Eq. (11), use is made of the geometry in Fig. 9.

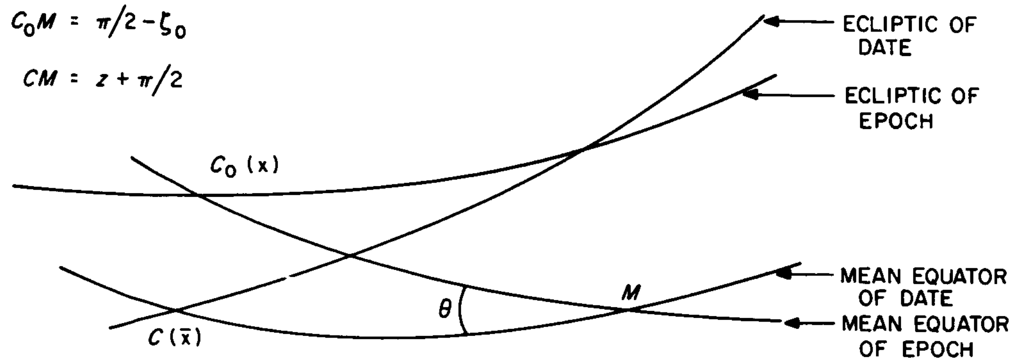

Fig. 9. Definition of mean equator and equinox of date and epoch

The lines in Fig. 9 are great circles on the surface of a sphere, with center at the center of the Moon. The positive x axis passes through the point C_0, and the $\bar{x}$ axis through C. The y axis lies in the C_0M plane, and the $\bar{y}$ axis in the CM plane. The notation ζ_0, z, θ, for the angles defined in the figure, is standard. Employing spherical trigonometry, it is easy to show that

$$\left.\begin{aligned}
\bar{a}_{11} &= -\sin\zeta_0 \sin z + \cos\zeta_0 \cos z \cos\theta \\
\bar{a}_{12} &= \sin\zeta_0 \cos z + \cos\zeta_0 \sin z \cos\theta \\
\bar{a}_{13} &= \cos\zeta_0 \sin\theta \\
\bar{a}_{21} &= -\cos\zeta_0 \sin z - \sin\zeta_0 \cos z \cos\theta \\
\bar{a}_{22} &= \cos\zeta_0 \cos z - \sin\zeta_0 \sin z \cos\theta \\
\bar{a}_{23} &= -\sin\zeta_0 \sin\theta \\
\bar{a}_{31} &= -\cos z \sin\theta \\
\bar{a}_{32} &= -\sin z \sin\theta \\
\bar{a}_{33} &= \cos\theta
\end{aligned}\right\} \tag{16}$$

Since the desired epoch is 1950.0, the quantities ζ_0, z, θ are given in the literature as

$$\left.\begin{aligned} \zeta_0 &= 0^\circ.64027694\,T + 0^\circ.838888(10^{-4})\,T^2 + 0^\circ.4972(10^{-5})\,T^3 \\ z &= 0^\circ.64027694\,T + 0^\circ.30361111(10^{-3})\,T^2 + 0^\circ.5333(10^{-5})\,T^3 \\ \theta &= 0^\circ.55674944\,T - 0^\circ.1183333(10^{-3})\,T^2 - 0^\circ.11555(10^{-4})\,T^3 \end{aligned}\right\} \tag{17}$$

where T is measured in Julian centuries (of 36525 days) from January 1.0,[4] 1950.

A simpler, and only slightly less accurate, method for computing the $\bar{a}_{ij}$ is

$$\begin{aligned} \bar{a}_{11} &= 1.00000000 - 0.29697(10^{-3})\,T^2 - 0.130(10^{-6})\,T^3 \\ \bar{a}_{21} = -\bar{a}_{12} &= -0.02234988\,T - 0.676(10^{-5})\,T^2 + 0.221(10^{-5})\,T^3 \\ \bar{a}_{31} = -\bar{a}_{13} &= -0.971711(10^{-2})\,T + 0.207(10^{-5})\,T^2 + 0.96(10^{-6})\,T^3 \\ \bar{a}_{22} &= 1.00000000 - 0.24976(10^{-3})\,T^2 - 0.15(10^{-6})\,T^3 \\ \bar{a}_{32} = \bar{a}_{23} &= -0.10859(10^{-3})\,T^2 - 0.3(10^{-7})\,T^3 \\ \bar{a}_{33} &= 1.00000000 - 0.4721(10^{-4})\,T^2 + 0.2(10^{-7})\,T^3 \end{aligned} \tag{18}$$

Here, T is measured in Julian centuries from January 1.0, 1950. (Eq. 17 and 18 appear in Ref. 1-3.)

Having now determined a', $\tilde{a}$, $\bar{a}$, the coefficients $a = (a_{ij})$ are then given by

$$a = (\bar{a}\,\tilde{a}\,a')^{-1} = (a')^{-1}\,(\tilde{a})^{-1}\,(\bar{a})^{-1} \tag{19}$$

[4] January 1.0 denotes January 1, 0 hr, UT, i.e., the midnight which ushers in the day, January 1, at Greenwich. This instant differs from 1950.0 by only a fraction of a mean solar day.

VI. FORMULAE FROM THE EPHEMERIS

From Eq. (13) and (15) we see that $a'_{ij} = a'_{ij}\,(\Omega', \Lambda, i)$ and, with the aid of (14), $\tilde{a}_{ij} = \tilde{a}_{ij}\,(\bar{\epsilon}, E, \psi)$. The defining expressions for these six quantities are lengthy and interrelated, and can be found among Ref. 1, 2 and 4. They are repeated here with modifications in notation, dimensions, and time scale.

$$\sin \Omega' = -\sin(\Omega + \sigma + \psi) \csc i \sin(l + \rho) \tag{20}$$

$$\cos i = \cos(l + \rho)\cos(\bar{\epsilon} + E) + \sin(l + \rho)\sin(\bar{\epsilon} + E)\cos(\Omega + \sigma + \psi) \tag{21}$$

$$\Lambda = \Delta + ☾ + \tau - \Omega - \sigma \tag{22}$$

$$\bar{\epsilon} = 23^\circ.4457874 - 0^\circ.01301376\,T - 0^\circ.8855(10^{-6})\,T^2 + 0^\circ.503(10^{-6})\,T^3 \tag{23}$$

$$E = \Delta\epsilon\ \text{(long period terms)} + d\epsilon\ \text{(short period terms)} \tag{24}$$

where

$$\begin{aligned}\Delta\epsilon = {} & 0^\circ.255833(10^{-2})\cos\Omega - 0^\circ.25(10^{-4})\cos 2\Omega \\ & + 0^\circ.1530555(10^{-3})\cos 2L + 0^\circ.61111(10^{-5})\cos(3L - \Gamma) \\ & - 0^\circ.25(10^{-5})\cos(L + \Gamma) - 0^\circ.194444(10^{-5})\cos(2L - \Omega) \\ & - 0^\circ.8333(10^{-6})\cos(2\Gamma' - \Omega)\end{aligned} \tag{25}$$

$$\left.\begin{aligned}d\epsilon = {} & 0^\circ.24444(10^{-4})\cos 2☾ + 0^\circ.5(10^{-5})\cos(2☾ - \Omega) \\ & + 0^\circ.30555(10^{-5})\cos(3☾ - \Gamma') - 0^\circ.13888(10^{-5})\cos(☾ + \Gamma') \\ & - 0^\circ.8333(10^{-6})\cos(☾ - \Gamma' + \Omega) + 0^\circ.8333(10^{-6})\cos(☾ - \Gamma' - \Omega) \\ & + 0^\circ.5555(10^{-6})\cos(3☾ - 2L + \Gamma') + 0^\circ.5555(10^{-6})\cos(3☾ - \Gamma' - \Omega)\end{aligned}\right\} \tag{26}$$

$$\psi = \Delta\psi \text{ (long period terms)} + d\psi \text{ (short period terms)} \tag{27}$$

where

$$\left.\begin{aligned}
\Delta\psi = & -[0^\circ.47895611(10^{-2}) + 0^\circ.47222(10^{-5})\,T]\ \sin\Omega \\
& + 0^\circ.58055(10^{-4})\sin 2\Omega - 0^\circ.35333(10^{-3})\sin 2L \\
& + 0^\circ.35(10^{-4})\sin(L-\Gamma) - 0^\circ.13888(10^{-4})\sin(3L-\Gamma) \\
& + 0^\circ.58333(10^{-5})\sin(L+\Gamma) + 0^\circ.3333(10^{-5})\sin(2L-\Omega) \\
& + 0^\circ.13888(10^{-5})\sin(2\Gamma'-\Omega) + 0^\circ.11111(10^{-5})\sin(2L-2\Gamma')
\end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned}
d\psi = & -0^\circ.56666(10^{-4})\sin 2\leftmoon + 0^\circ.18888(10^{-4})\sin(\leftmoon-\Gamma') \\
& +0^\circ.83333(10^{-6})\sin 2(\leftmoon-\Gamma') - 0^\circ.94444(10^{-5})\sin(2\leftmoon-\Omega) \\
& -0^\circ.7222(10^{-5})\sin(3\leftmoon-\Gamma') + 0^\circ.41666(10^{-5})\sin(\leftmoon-2L+\Gamma') \\
& +0^\circ.30555(10^{-5})\sin(\leftmoon+\Gamma') + 0^\circ.16666(10^{-5})\sin 2(\leftmoon-L) \\
& +0^\circ.16666(10^{-5})\sin(\leftmoon-\Gamma'+\Omega) + 0^\circ.16666(10^{-5})\sin(\leftmoon-\Gamma'-\Omega) \\
& -0^\circ.13888(10^{-5})\sin(3\leftmoon-2L+\Gamma') - 0^\circ.1111(10^{-5})\sin(3\leftmoon-\Gamma'-\Omega)
\end{aligned}\right\} \tag{29}$$

The remaining quantities are

$$I = 1^\circ\,32.1' \tag{30}$$

$$\Omega = 12^\circ.1127902 - 0^\circ.0529539222\,d + 0^\circ.20795(10^{-2})\,T + 0^\circ.2081(10^{-2})\,T^2 + 0^\circ.2(10^{-5})\,T^3 \tag{31}$$

$$\leftmoon = 64^\circ.37545167 + 13^\circ.1763965268\,d - 0^\circ.1131575(10^{-2})\,T - 0^\circ.113015(10^{-2})\,T^2 + 0^\circ.19(10^{-5})\,T^3 \tag{32}$$

$$\left.\begin{aligned}
\sin\Delta &= -\sin(\Omega+\sigma+\psi)\csc i\sin(\bar{\epsilon}+E), \\
\cos\Delta &= -\cos(\Omega+\sigma+\psi)\cos\Omega' - \sin(\Omega+\sigma+\psi)\sin\Omega'\cos(\bar{\epsilon}+E)
\end{aligned}\right\} \tag{33}$$

$$\Gamma' = 208^\circ.8439877 + 0^\circ.1114040803\, d - 0^\circ.010334\, T - 0^\circ.010343\, T^2 - 0^\circ.12(10^{-4})\, T^3 \quad (34)$$

$$L = 280^\circ.08121009 + 0^\circ.9856473354\, d + 0^\circ.302(10^{-3})\, T + 0^\circ.302(10^{-3})\, T^2 \quad (35)$$

$$\Gamma = 282^\circ.08053028 + 0^\circ.470684(10^{-4})\, d + 0^\circ.45525(10^{-3})\, T + 0^\circ.4575(10^{-3})\, T^2 + 0^\circ.3(10^{-5})\, T^3 \quad (36)$$

In Eq. (23), (31), (32), (34), (35), and (36), T is measured in Julian centuries (of 36525 days) from January 1.0, 1950, and d is measured in Julian days from this same date. The quantities I, Ω, ☾, and Δ are illustrated in Fig. 10.

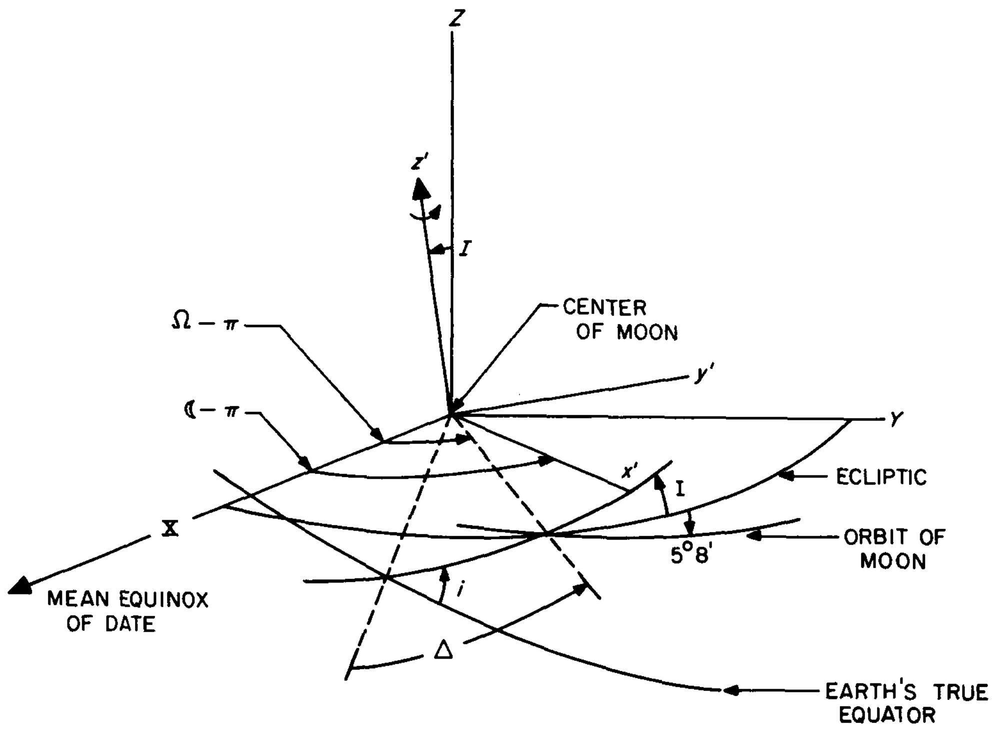

Fig. 10. Definition of I, Ω, ☾, and Δ

The symbol Ω is the longitude of the mean ascending node of the Moon's orbit measured in the ecliptic (X Y plane) from the mean equinox of date ($+X$ axis); ☾ is the mean longitude of the Moon measured in the ecliptic from the $+X$ axis to the mean ascending node of the Moon's orbit, and then along the orbit; Γ' is the mean longitude of the

Moon's perigee, measured the same way as ☾. The quantities L and Γ are the mean longitude of the Sun and of the Sun's perigee, respectively. ($L - \Gamma$ is the mean anomaly of the Sun.)

The quantities σ, τ, ρ are small perturbations in Ω, ☾, I, respectively. (Refer to Eq. 20, 21, 22 and 33.) They appear in Ref. 4, and are repeated here with an altered time scale.

$$\sigma \sin I = -0^\circ.0302777 \sin g + 0^\circ.0102777 \sin (g + 2\omega) - 0^\circ.305555(10^{-2}) \sin (2g + 2\omega) \tag{37}$$

$$\tau = -0^\circ.3333(10^{-2}) \sin g + 0^\circ.0163888 \sin g' + 0^\circ.5(10^{-2}) \sin 2\omega \tag{38}$$

$$\rho = -0^\circ.0297222 \cos g + 0^\circ.0102777 \cos (g + 2\omega) - 0^\circ.305555(10^{-2}) \cos (2g + 2\omega) \tag{39}$$

where

$$g = 215^\circ.54013 + 13^\circ.064992\, d \tag{40}$$

$$g' = 358^\circ.009067 + 0^\circ.9856005\, d \tag{41}$$

$$\omega = 196^\circ.745632 + 0^\circ.1643586\, d \tag{42}$$

Here, d is measured in Julian days from January 1.0, 1950.

VII. SIMPLIFICATION

A large amount of lunar exploration can be expected in the coming decade. During the period 1960 - 70, T will lie in the range

$$0.10 \leq T \leq 0.20$$

so that terms involving T^3 in Eq. (17), (18), (23), (31), (32), (34), (35), (36) can, for all practical purposes, be omitted.

An important quantity to know is the velocity of the probe relative to the Moon-fixed system; i.e., $v_1' = (\dot{x}_1'^2 + \dot{y}_1'^2 + \dot{z}_1'^2)^{1/2}$. The velocity components $\dot{x}_1'$, $\dot{y}_1'$, $\dot{z}_1'$ are obtained from Eq. (12) and (19):

$$\frac{dX'}{dt} = \frac{da}{dt} X + a \frac{dX}{dt} \tag{43}$$

where

$$\begin{aligned} \frac{da}{dt} &= \frac{d}{dt} (a')^{-1} \cdot (\tilde{a})^{-1} \cdot (\bar{a})^{-1} + (a')^{-1} \cdot \frac{d}{dt} (\tilde{a})^{-1} \cdot (\bar{a})^{-1} + (a')^{-1} \cdot (\tilde{a})^{-1} \cdot \frac{d}{dt} (\bar{a})^{-1} \\ &= \left(\frac{da'}{dt}\right)^{-1} \cdot (\tilde{a})^{-1} \cdot (\bar{a})^{-1} + (a')^{-1} \cdot \left(\frac{d\tilde{a}}{dt}\right)^{-1} \cdot (\bar{a})^{-1} + (a')^{-1} \cdot (\tilde{a})^{-1} \cdot \left(\frac{d\bar{a}}{dt}\right)^{-1} \end{aligned} \tag{44}$$

It is evident from Eq. (13), (15), (18), (20) - (42) that the elements of a', $\tilde{a}$, $\bar{a}$ are ultimately a function of T, T^2, and d. Since the desirable units of v_1' are distance per *second*, the time derivative of these elements will be a function of

$$\frac{dT}{dt} = \frac{1}{p}, \qquad \frac{dT^2}{dt} = \frac{2t}{p^2}, \qquad \text{and} \qquad \frac{d(d)}{dt} = \frac{1}{q}$$

where $p = 60 \times 60 \times 24 \times 36525$ and $q = p/36525$. Here, t is measured in seconds from January 1.0, 1950. For the decade 1960 - 70,

$$3.15576(10^8) \leq t \leq 6.31152(10^8) \text{ sec} \tag{45}$$

Since the elements of $\bar{a}$ are explicit functions of T and T^2 (refer to Eq. 18), it is easily shown that

$$\left|\frac{d\bar{a}_{ij}}{dt}\right| < 0.7(10^{-11}) \text{ sec}^{-1} \tag{46}$$

The elements of $\tilde{a}$ are a function of $\bar{\epsilon}$, E, and ψ. It can be shown that

$$\left|\frac{d\bar{\epsilon}}{dt}\right| < 0.8(10^{-13}) \text{ sec}^{-1}, \qquad \left|\frac{dE}{dt}\right| < 0.3(10^{-11}) \text{ sec}^{-1}, \qquad \left|\frac{d\psi}{dt}\right| < 0.3(10^{-10}) \text{ sec}^{-1}$$

In view of Eq. (15),

$$\left|\frac{d\tilde{a}_{ij}}{dt}\right| < 0.3(10^{-10}) \text{ sec}^{-1} \tag{47}$$

The elements of a' are a function of Ω', i, and Λ, where Λ is given by Eq. (22). From page 51 of Ref. 2,

$$\left|\frac{d\Omega'}{dt}\right| < 0.7(10^{-9}) \text{ sec}^{-1}, \qquad \left|\frac{di}{dt}\right| < 0.82(10^{-9}) \text{ sec}^{-1}, \qquad \left|\frac{d\Delta}{dt}\right| < 0.102(10^{-7}) \text{ sec}^{-1}$$

Computing the limits on the time derivatives of $☾$, τ, Ω, σ from Eq. (32), (38), (31), (37), respectively, it follows that $|d\Lambda/dt| < 0.3(10^{-5}) \text{ sec}^{-1}$. Consequently, from Eq. (13),

$$\left|\frac{da'_{ij}}{dt}\right| < 0.3(10^{-5})\ \sec^{-1} \tag{48}$$

We write Eq. (44) as $da/dt = K + L + M$, where K, L, M correspond, respectively, to the terms in Eq. (44). Since each of the third-order matrices a', $\tilde{a}$, $\bar{a}$ is orthogonal, the maximum absolute value of any element of the product matrices $(\tilde{a})^{-1} \cdot (\bar{a})^{-1}$, $(a')^{-1} \cdot (\tilde{a})^{-1}$ is unity. Hence,

$$|K_{ij}| < 3 \left|\frac{da'_{ij}}{dt}\right|_{max} = 3(.3)(10^{-5})\ \sec^{-1},\quad |L_{ij}| < 9(.3)(10^{-10})\ \sec^{-1},\quad |M_{ij}| < 3(.7)(10^{-11})\ \sec^{-1} \tag{49}$$

Consider the first term in Eq. (43). Let

$$\frac{da}{dt}\,X = (K + L + M)\,X = K\,X + L\,X + M\,X = U + V + W \tag{50}$$

The largest distance from the Moon at which v'_1 will be of practical interest is approximately 10 lunar radii[5], i.e., 17,380 km. We let the maximum absolute value of any element of X equal this distance. Then, from (49), it follows that

$$|U_{ij}| < 3\,|K_{ij}|_{max}\ (17{,}380\text{ km}) = 470\text{ meters/sec}$$

$$|W_{ij}| < 0.0141\text{ meters/sec}$$

$$|W_{ij}| < 0.109(10^{-2})\text{ meters/sec}$$

In view of the upper bounds on $|V_{ij}|$ and $|W_{ij}|$, little accuracy will be lost if V and W are considered negligible in Eq. (50). Hence, we shall write Eq. (44) as

---

[5] The orbital radius of a lunar satellite will probably not exceed 10 lunar radii.

$$\frac{da}{dt} \sim \left(\frac{da'}{dt}\right)^{-1} \cdot (\tilde{a})^{-1} \cdot (\bar{a})^{-1} \quad . \tag{44a}$$

A further simplification can be made in the time derivative of a'. Since the a'_{ij} are functions of Ω', i, Λ, the elements of da'/dt will be functions of the time derivatives of these same quantities. In view of the upper bounds on $|d\Omega'/dt|$, $|di/dt|$, $|d\Lambda/dt|$, given above, terms which contain $d\Omega'/dt$ or di/dt can, for all practical purposes, be ignored. The elements of da'/dt are then given by the expressions below.

$$\left.\begin{aligned}
\frac{da'_{11}}{dt} &\sim -\cos\Omega' \sin\Lambda \cdot \frac{d\Lambda}{dt} - \sin\Omega' \cos i \cos\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{21}}{dt} &\sim -\sin\Omega' \sin\Lambda \cdot \frac{d\Lambda}{dt} + \cos\Omega' \cos i \cos\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{31}}{dt} &\sim +\sin i \cos\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{12}}{dt} &\sim -\cos\Omega' \cos\Lambda \cdot \frac{d\Lambda}{dt} + \sin\Omega' \cos i \sin\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{22}}{dt} &\sim -\sin\Omega' \cos\Lambda \cdot \frac{d\Lambda}{dt} - \cos\Omega' \cos i \sin\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{32}}{dt} &\sim -\sin i \sin\Lambda \cdot \frac{d\Lambda}{dt} \\
\frac{da'_{13}}{dt} &\sim 0, \qquad \frac{da'_{23}}{dt} \sim 0, \qquad \frac{da'_{33}}{dt} \sim 0
\end{aligned}\right\} \tag{51}$$

where

$$\frac{d\Lambda}{dt} = \frac{d(\Delta)}{dt} + \frac{d☾}{dt} + \frac{d\tau}{dt} - \frac{d\Omega}{dt} - \frac{d\sigma}{dt}$$

Here,

$$\frac{d\Delta}{dt} \sim \frac{-\sin(\bar{\epsilon} + E)\cos(\Omega + \sigma + \psi)\cdot\left(\frac{d\Omega}{dt} + \frac{d\sigma}{dt}\right)}{\sin i \cos\Delta} \text{ rad/sec} \tag{52}$$

$$\frac{d☾}{dt} = 0.266170762(10^{-5}) \text{ rad/sec} - 0.39607482(10^{-23})\, t \text{ rad/sec} \tag{53}$$

$$\frac{d\tau}{dt} = -0.153527294(10^{-9}) \cos g + 0.569494067(10^{-10}) \cos g' + 0.5794733484(10^{-11}) \cos 2\omega \text{ rad/sec} \tag{54}$$

$$\frac{d\Omega}{dt} = -0.106969843(10^{-7}) \text{ rad/sec} + 0.729311779(10^{-23})\, t \text{ rad/sec} \tag{55}$$

$$\frac{d\sigma}{dt} = -0.520642191(10^{-7}) \cos g + 0.181177445(10^{-7}) \cos(g + 2\omega) - 0.106405785(10^{-7}) \cos(2g + 2\omega) \text{ rad/sec} \tag{56}$$

where g, g', ω are given by Eq. (40) – (42). In Eq. (53) and (55), t is measured in seconds from January 1.0, 1950.

VIII. PRINCIPAL MOMENTS OF INERTIA

The values of A, B, C obtained from the literature are computed on the basis of observations made of the Moon's rotary and orbital motion. Thus, the ratio $(C - A)/C$ depends on the orientation of the Moon's rotation axis, while the quantity $(3/2)[(B-A)/MR^2]$ depends on the Moon's annual libration in longitude (λ_e). Here, M = mass of Moon and R = mean radius of the Moon, as measured from the Earth. In addition, the quantity $(3/2)(C/MR^2)$ is a measure of the Moon's density distribution, which is ascertained from the Moon's orbital motion. Values of these three quantities are taken from Ref. 5 (pp. 158, 164, 166, and 168).

$$\left.\begin{aligned} \frac{C-A}{C} &= 0.629(10^{-3}) \\ \frac{3}{2}\,\frac{B-A}{MR^2} &= 0.59(10^{-4}) \\ \frac{3}{2}\,\frac{C}{MR^2} &= 0.5956 \end{aligned}\right\} \tag{57}$$

If values are assigned to M and R, then Eq. (57) can be solved for A, B, C. From Ref. 6 (p. 156), $G = 0.6673(10^{-10})$ meters^3/kg sec^2. From values of M_e/M (M_e = mass of Earth), R_e (= equatorial radius of Earth), 1 foot/international meter,and GM_e (Earth radii)^3/min^2, Ref. 7, one computes $GM = 0.48984463(10^{13})$ meters^3/sec^2. Combining this with the above value of G, we get $M = 0.73406957(10^{23})$ kg. Now,

$$R = s\,D$$

$$R_e = (H.\ P.)\,D$$

where D is the distance between the centers of Earth and Moon, s is the Moon's semidiameter (angle subtended at the Earth's center by R) and $H.\ P.$ is the Moon's horizontal parallax (angle subtended at the Moon's center by R_e). Eliminating D gives

$$R = \frac{s}{H.\ P.} R_e$$

where, $R_e = 0.637827(10^7)$ meters. Values of s and $H.\ P.$ for every half day are listed in Ref. 2 (pp. 52 – 67). Eight sets of values of s and $H.\ P.$ were selected which corresponded to four locations of the Moon at apogee and four at perigee, in the months of January, February, November, and December. Substituting these in the above formula yielded eight values of R which differed at most by 0.034793 km. We adopt the following representative value of R: $R = 1737.880$ km. Substituting M and R in Eq. (57) yields

$$\begin{aligned} A &= 0.8797655(10^{35}) \text{ kg meters}^2 \\ B &= 0.8798527(10^{35}) \text{ kg meters}^2 \\ C &= 0.8803192(10^{35}) \text{ kg meters}^2 \end{aligned} \tag{58}$$

We can look forward to the time when extended observations of an artificial lunar satellite will yield significantly more accurate values of A, B, C.

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REFERENCES

1. *The Nautical Almanac,* U. S. Government Printing Office, Washington, D. C., 1941.

2. *The American Ephemeris and Nautical Almanac,* U. S. Government Printing Office, Washington, D. C., 1960, 1961.

3. *Planetary Coordinates for the Years 1960-1980,* Her Majesty's Stationery Office, London, 1958.

4. Hayn, F., "Abhandlung der K. S. Gesellschaft der Wissenschaften," Mathematical Physics, *Selenographische Koordinaten,* Vol. 27, 29, 30 (1902, 1906, 1909).

5. Jeffreys, H., *The Earth,* 4th ed., Cambridge University Press, 1959.

6. Heiskanev, W. A., and Vening Meinesz, F. A., *The Earth and Its Gravity Field,* McGraw-Hill Book Company, Inc., New York, 1959.

7. Herrick, S., Baker, R. M. L., et al, *Gravitational and Related Constants for Accurate Space Navigation,* Astronomical Paper 24, University of California at Los Angeles, Astronomy Department, pp. 297-338.